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ABSTRACT

In Japan, assessment in classroom teaching cannot be considered apart from classroom lessons. This paper describes typical mathematics classroom situations and some of the developments in classroom assessment underway in Japan. The first section of the paper discusses classroom teaching, including whole-class instruction, discussion, objectives, and the importance of lesson plans and records. Next, assessment practices during and after lessons are discussed. Two long-term research projects to assess higher-order thinking are then considered. The Open-End Approach to Teaching Mathematics Project included three types of open-ended mathematics problems: how to find rules or relations, how to classify, and how to measure. Students' achievement was evaluated in terms of fluency, flexibility, and originality. The Developmental Treatment of Mathematics Problems Project had features similar to the Open-End project, except that students formulated or posed problems of their own. Student evaluation was conducted from three points of view: by the number of problems, by how the students formulated their problems, and by the direction of development from the given problem. Samples of lesson plans, classroom dialogue, and sample problems formulated by both teachers and students are provided throughout the document. Contains 25 references. (MKR)

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CLASSROOM ASSESSMENT IN JAPANESE MATHEMATICS EDUCATION*

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*This article appeared in condensed form in NCTM's 1993 Yearbook Assessment in the Mathematics Classroom, (pp. 40-53), edited by Norman L. Webb.

This article was a fully and equally shared effort between the two authors. It stems from the second author's interest in Japanese mathematics education, dating back many years, the U.S. - Japan Seminar on Mathematical Problem Solving (Becker and Miwa 1987), and his professional interaction with the first author and many of his colleagues. The article follows on the heels of the completion of the translation of the book on the "open end approach," which was edited by Professor Shigeru Shimada. The authors express their appreciation to Professor Shimada for his very helpful comments to a draft of the paper and also to Norman Webb for his very useful suggestions and patience in completing the task.

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I. INTRODUCTION

For several decades, assessment (evaluation) has been discussed from various points of view in Japanese mathematics education. For example, the following questions have been discussed:

> How are the grading (rating) and assessment (evaluation) of student performance related?
>
> How are interest in, and attitudes toward mathematics assessed?
>
> Should we use criterion-referenced or norm-referenced assessment for grading purposes?
>
> How can we cope with the effects of the entrance examinations as a part of external assessment?

Sometimes the media (e.g., newspapers and magazines) have functioned as a forum in the debate about the merit(s) or demerit(s) of assessment, especially with respect to the effects of the entrance examinations. Indeed, ordinary people have also been involved in the controversary surrounding the entrance examinations. Also, more recently criterion-referenced assessments has become an issue along with the question of how to assess (*a)* ways of mathematical thinking and (*b*) interest and attitudes toward mathematics. Both entrance examinations and criterion-referenced assessment have become practical problems with which Japanese mathematics educators have been struggling, with emphasis on both (*a*) and (*b*).

During the past couple of decades, the idea of formative evaluation has gained prominence in Japanese education in general and in mathematics education in particular.

However, the idea has been recognized and discussed primarily in an academic circle of educators and has not been implemented on a large scale in the schools with respect to either learning or teaching. In reality, summative evaluation (Bloom 1971) has been the main approach to evaluation in Japan. Mathematics educators, teachers, and researchers are grappling with these ideas of evaluation, and the situation is gradually changing more at the elementary school level than at the lower and upper secondary school levels.

In this article, we shall describe some of the efforts in classroom assessment that have been under way in Japan for some time now. Before doing so, we first want to describe the typical classroom situation in mathematics, since assessment is considered to be an integral part of classroom teaching. Furthermore, what takes place in the Japanese classroom is somewhat different from that in the United States and some European countries.

II. THE MATHEMATICS CLASSROOM IN JAPANESE SCHOOLS

1. Classroom Teaching

Whole class instruction is the approach used by mathematics teachers in Japanese elementary and secondary school classrooms. All classrooms are equipped with a large chalkboard on the front wall, many with an overhead projector and screen, and some teachers also use small 2′ x 2′ or 3′ x 3′ chalkboards (which are hung from the top of the large chalkboard) or poster boards attached to chalkboards by magnets . Teachers use the objects for the presentation of problems and solutions, or at times the teacher has students write their problem solutions and approaches on them for display to the whole class.

Class size in Japan is much larger than in the United States. Typically there are 30-45 students in a class at the elementary and secondary levels. They sit in a boy-girl configuration at desks with benches (sometimes in rows of single desks). Students, especially in elementary schools, are quite disciplined and attentive during class and also

somewhat formal compared with their cohorts in the United States. For example, a lesson begins and ends with students rising and bowing to the teacher, and the teacher reciprocates. A similar situation prevails in high schools, though at present some high schools are experiencing discipline problems with students. Since there are ten minutes between classes, students have an opportunity to relax and "unwind" after intensive concentration during class; at times, the teacher may extend the class period in order to complete and polish the lesson.

Generally the teacher develops the lesson around one single objective (e.g., a topic or behavioral objective), and class activities are focused on it. The main role of the teacher is that of a guide, not a "dispenser of knowledge." The activities and sequence of events in a lesson are commonly organized to draw out the variety of student's thinking, and the teacher's "wait time" is crucial in this respect. The different ways students think about the mathematical topic or problem in a lesson are respected to a very significant degree by the teacher; in fact, the dynamics of a lesson center on this, and teachers rely on students as an "information source" during the lesson. The discussion of students' ideas is also a prominent characteristic. Similarly, students are expected to give verbal explanations, sometimes lengthy ones, of their ideas (cf. Stigler 1988). Toward the end of the lesson, the teacher "pulls together" students' ways of thinking, discusses their mathematical quality, and then summarizes, elaborates, or "polishes up" the lesson. Discussion (whole class or small group) among students or between the teacher and students is extensive and is a major factor in achieving the lesson's objective (cf. Becker et al. 1989, Becker et al. 1990, Stigler 1988, Stigler & Stevenson 1991 and Miwa 1992).

Lessons are also intensive. The lesson moves toward the objective with minimal external interruptions (e.g., public address announcements or students entering or leaving the room). There is a certain discipline about this, in which students' own responsibility in

learning is reflected. Boy-girl interaction is common, and teachers have high expectations of both sexes.

What goes on in the teachers' rooms is another important characteristic of Japanese education. Unlike in the United States, teachers in Japan share a large room, each with a desk, chair, and file. This arrangement is no trivial detail, for it gives teachers an opportunity to interact: they plan lessons, discuss written records of teaching, discuss and plan evaluation, and, in general, discuss professional matters relating to their students, teaching, and the mathematics curriculum.

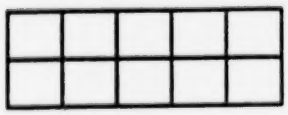
2. The Role of Lesson Plans

The philosophy and reality of lesson plans and lesson records of teaching in Japan are considerably different from those in the United States or some European countries. Regarding lesson plans, many American teachers may think it impossible to anticipate students' responses in detail; therefore, their lesson plans may be somewhat rough or not detailed. Many Japanese teachers, however, think that it is crucially important to develop and polish lesson plans in a collaborative manner, including listing student's anticipated responses to the problems posed in a lesson (see figure 5.1). In this way teachers get a better understanding of a lesson themselves, and they are better prepared to anticipate and deal with students' responses and viewpoints in the actual teaching. A typical Japanese classroom lesson may be compared to a drama, with the lesson plan the script (see Yoshida 1992).

Objective: To help students solve the problem by drawing out students' natural ways of thinking, comparing them, and finding a rule.

Teaching

1. The problem:



Problem

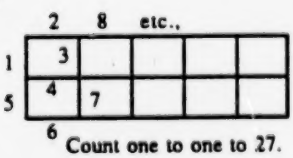
If matchsticks touch only at their endpoints as in he figure, how many matchsticks do we need to make 10 squares? Find the answer in as many ways as possible.

2. Introducing the problem:

Teacher: How many matchsticks do we need when we have two squares?
Student: Seven matchsticks
Teacher: What about when we have four squares?
Student: Twelve matchsticks
Teacher: Now find the answer to the problem in as many ways as possible. Think out a variety of ways for determining the number of matchsticks. Show you work on your worksheet.

3. Students' anticipated responses:

(a)



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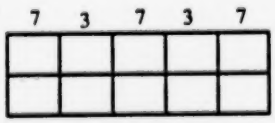

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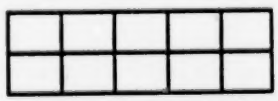

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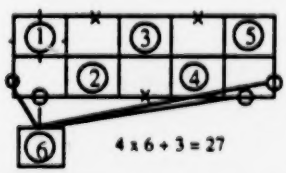

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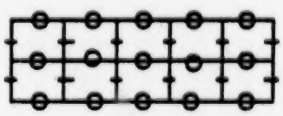

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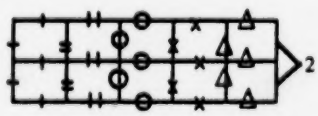

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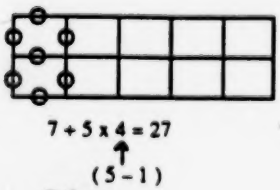

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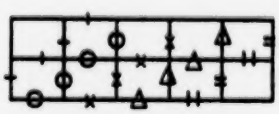
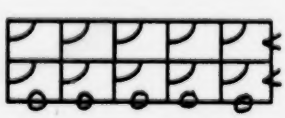


(i)



(j) Others

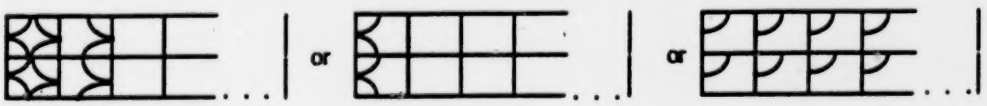
Note: "Two times three" is written 3 x 2 in Japan.

4. Discussing each way with students and classifying them according to some shared feature. Compare the different ways according to their mathematical quality.

   (a) Which way do you think is best? Why?

   (b) What happens when the number of squares increases? Explain.

   (c) Which is the easiest way when we have 20 squares?



$7 + 5 \times (\frac{20}{2} - 1) = 52$ $\quad 5 \times \frac{20}{2} + 2 = 52$ $\quad 2 \times 20 \div \frac{20}{2} + 2 = 52$

5. Generalize:
   $7 + 5 \times (n \div 2 - 1) = \#$ of matchsticks
   $5 \times (n \div 2) + 2 = \#$ of matchsticks
   $2 \times n + (n \div 2) + 2 = \#$ of matchsticks

6. Summing up

7. Homework: How many matchsticks do we need when we have the following shapes?

   (a) 7 squares
   (b) 15 squares

Figure 5.1 **A teacher's plan for a lesson on problem solving in grade 6 (edited from the Japanese).**

III. A PERSPECTIVE ON ASSESSMENT ON JAPAN

1. Approach to Assessment

Within the framework of whole-class instruction, many Japanese teachers respect classroom teaching that is directed toward using different ways of student's thinking in order to raise the level of the understanding of the class as a whole. Therefore, the focus of assessment is on "how each student thinks according to her or his natural way of thinking or ability." These ways of thinking mathematically are regarded as concrete information about students' progress in learning. Both correct and incorrect ways of students' thinking are naturally included. It is believed that using all the ways in a lesson helps to enhance students' learning.

In this approach, teachers' observations of students during the lesson are an important source of information for assessment. According to their observations, teachers can adjust their teaching to cope with, for example, the following matters:

- To see how well students understand their task
- To select which response(s) will be presented to the whole class
- To enhance the quality of discussion
- To pay attention to students' individual needs

There are two types of observations: observations of students' work on the problem while walking around the room, and observations made during discussions with students. Included in the first type is observing whether students' responses are as anticipated. After the lesson, students' worksheets are collected and analyzed as another crucial source of information with respect to an evaluation both of the lesson and of individual students' performance.

This approach to assessment is accepted as important by many teachers at the elementary school level; the higher the grade level, however, the fewer the number of

teachers who actually use this approach. One main reason is that they do not have enough time for this approach to assessment. Another reason, especially for high school teachers, is the importance of, and the time devoted to preparation for, entrance examinations (at grades 9 and 12). Therefore, summative evaluation, which depends on paper-and-pencil tests, is generally the main approach in classrooms at all school levels. This trend is especially strong in senior high schools.

Now, how is the formative approach explicitly realized in practice? In Japan, in-service teacher education is carried out in each school, in education centers, and in private study groups. Many such experiences are classroom and research-based in that teachers develop a lesson plan cooperatively, then one teacher (a representative of the group) teaches the lesson while the other teachers observe the lesson in progress, and afterward a record of the lesson is written and the teachers discuss it. Of course, the aim is not summative evaluation; rather, it is formative. Usually, such meetings are held once a month or once a term. Sometimes it may take several months to plan, implement, and analyze a lesson in a thorough fashion. Even if a teacher participates in this type of cooperative activity only once a year, the teacher can learn to understand and appreciate the process and to reflect on it with respect to her or his own approach to teaching.

To reinforce this in-service education, several monthly journals for mathematics teachers include articles about this approach to developing lesson plans and lesson records, and teachers have easy access to them. It is also noteworthy that more journals on mathematics education are available for elementary teachers than for high school teachers. For example, there is Arithmetic Education, published by the Japan Society of Mathematics Education (JSME), and four commercial journals all published nationwide.

Especially at the elementary school level, teachers understand that classroom assessment needs to be integrated into classroom lessons. This may lead, in a natural way, to curriculum improvement based on classroom practice. In Japan, this concept is expressed by the slogan We Should Learn from the Students.

The Japanese educational system is more centralized than that of the United States, but the approach described here has been established as a tradition, and it permits a "bottom up" approach to improvement with teachers as agents of change. This is in contrast to a "top down" approach in which teachers are regarded as targets of change. Making teachers agents rather than targets of change seems more desirable.

2. Assessment Practice

In this section, assessment both during and after lessons is considered. Several lesson records made by teachers after they taught the lessons are used to show some aspects of assessment. Here we focus on lessons in which the teachers try to use students' different methods of solving a problem, which reflect their ways of thinking, to evaluate their students' learning.

(1) Assessment During lessons

Classroom lessons mainly involve whole-class instruction, as mentioned earlier, and the chief means of assessment is teacher observation. Here we see from lesson records how assessment during lessons is implemented. It is important to observe how students think. In this stage, an assessment of classroom instruction is the main objective but if observations of each student are accumulated, the results of the assessment will become information for the summative evaluation of each student.

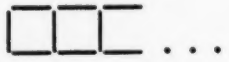
a) *Assessment of concept formation in the whole class*

In whole-class instruction, students' different ways of thinking can be used to form a concept from different points of view. It is crucial, however, first to formulate a problem situation in which every student can have some success in finding some solution methods(s). After students exhibit their ways of thinking, the teacher should classify them to form a concept.

Example - Number of matches: fifth grade (Hashimoto 1989)

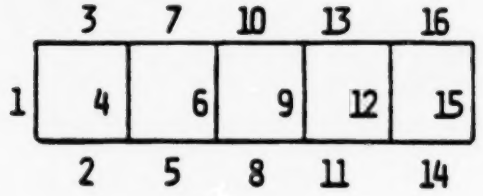
The teacher, Mr. Tsubota, presented the following problem:

Squares are made using matches as shown in the figure below. When the number of squares is 5, how many matches are used?

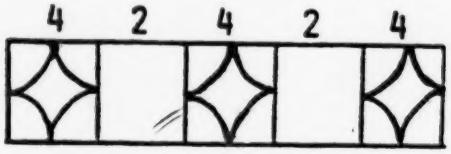


Students presented their methods of finding the answer as follows:

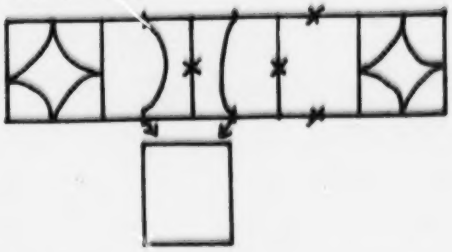
- Drawing the figure and counting one by one (important for less able students)



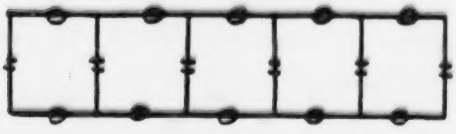
- $4 \times 3 + 2 \times 2 = 16$



- $4 \times 4 = 16$



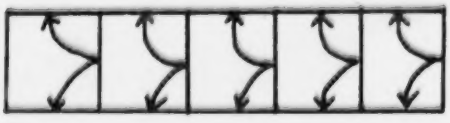
- $2 \times 5 + 6 = 16$



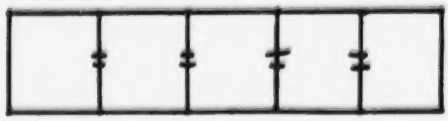
- $3 \times 5 + 1 = 16$

- $4 + 3 \times 4 = 16$

- $3 \times 5 + 1 = 16$



- $4 \times 5 - 4 = 16$



The teacher could see that students understood this problem very well, for they developed numerous methods for finding the answer. Since students looked at this problem from several viewpoints, the teacher could now proceed to the next stage, namely, asking students to make up, formulate, or pose new problems by themselves. So the next lesson began with a review of the last lesson, including reference to the eight ways students used to get the answer. Excerpts of the remainder of the record of the whole lesson follow ["___-kun" denotes boys, "___-san" denotes girls]:

T: In today's lesson, I won't pose a problem, but you will pose one by making up a problem similar to the one you just solved. I want you to present the problem you made yourself and discuss it with one another.

S: Is it okay if it's only a little bit similar?

T: Yes, it's okay.

S: Is it okay if we use triangles or pentagons instead of squares?

T: That's a good idea, but if you say your ideas out loud, others may end up using them. Let's begin.

S: Teacher, may I draw a figure?

T: Yes, if you draw a figure it will make it easy for others to understand your problem.

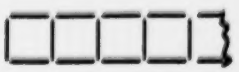
(Teacher walks around, scanning student's work)

... .

T: Sonobe-kun, please come up and explain your problem.

S: I changed the first problem a little and made this problem:

> Squares are made using iron sticks. If the number of squares is 30, how many iron sticks are used?



T: What is the length of all the sticks? Are they the same length?

S: Constant length sticks.

... .

T: Did anyone make a problem similar to this? Shoji-kun?

S: My number is different. Seventy sticks.

T: How many people changed the number of squares?

[10 children raised their hands.]

... .

T: Tani-kun, please explain your problem. Listen to his idea, everyone.

S: Squares are made by matches in the first problem. I made the problem different by changing squares to equilateral triangles like this:

> Equilateral triangles are made by using matches as shown in the figure. When the number of equilateral triangles is 15, how many matches are used?

T: Did you only change squares to equilateral triangles?

S: I also changed the number.

T: Raise your hand if you changed squares to triangles.

[Many children raised their hands.]

T: Oh, so many. Well, how many people changed squares to geometrical figures other than triangles?

T: What figures did you make, Endo-kun? Come up and put yours on the blackboard.

S: Well, I changed squares in the first problem to regular hexagons, and I changed the number from 5 to 1011:

> Matches are arranged as shown in the figure. When the number of regular hexagons is 1011, how many matches are used? (Use matches of all the same length.)

T: Can you solve it? I think you can.

S: It's solvable if I compute it. Maybe I can.

... .

T: There are two types so far-those formed by changing squares to equilateral triangles and those formed by changing squares to regular hexagons.

T: Did anyone pose the problem by changing to other figures besides the hexagon? What did Suzuki-san do?

S: I want to make four pentagons with five beads per side. How many beads are used?

T: Please draw your figure. By hand is okay.

... .

T: Triangle, hexagon, pentagon. Did anyone make other figures? Yes, Kosaka-san.

S: Rectangular solid.

T: Rectangular solid? Did you draw it? That's interesting.

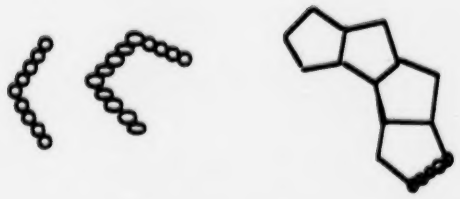
T: Tsunashima-kun?

S: Rectangle.

T: Draw your figure.

T: Suzuki-kun? Write your idea above Suzuki-san's response.

[Suzuki-kun drew the two left figures first, and then moved to Suzuki-san's response in the right figure.]



T: You just fill in one side of beads with yellow chalk, we can understand. Please explain.

S: I want to make four pentagons with 5 beads per side. How many beads are used?

... .

T: She made such a problem...the figure is a pentagon. Thanks. Any questions? Ariga-kun, okay?

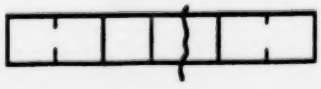
S: Yes. A vertical figure of regular pentagons is made using matches. When the number of regular pentagons is 726, how many matches are used?

T: Regular pentagons are connected like this. It's different from the first problem, because in his problem, the figure is zigzag, while the first one isn't. Thanks.

... .

T: Then Tsunashima-kun. What is your problem?

S:



Rectangles and squares are made using matches as in the figure. When the number of rectangles and squares altogether is 1111, how many matches are used? [One side of the rectangle is two times that of the square.]

T: I don't understand the meaning of the figure.

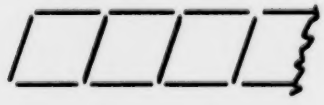
S: I know. Rectangle, square, rectangle and square. [Tsunashima-kun calls on Ariga-kun, who has raised his hand.] If one side is doubled, is each side of the rectangle doubled? Are both width and height doubled?

S: Only the width is doubled.

T: Well, one more person, Suzuki-kun, come up with a different way of posing the problem.

S: Yes. I almost completely changed the problem. And this is the problem.

> Parallelograms are made by using pencils of the same length as shown in the figure below. When the number of pencils is 37, how many parallelograms can be made and how many pencils will remain?



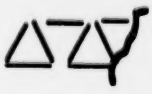
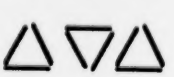
T: How did you change it? You said you almost completely changed it.

S: Yes. I changed matches to pencils and squares to parallelograms. And instead of asking how many matches make squares, I asked how many parallelograms can be made with 37 pencils and how many pencils will remain.

T: It seems a little difficult, but any questions?

S: What if there is no remainder? It's okay.

. . .

[There ensues a discussion of what a square is, so it isn't a square (no 90° angles). It's a rhombus, since the pencils are the same length, but it is also a parallelogram. There is also discussion of division in finding the answer. Then there was a fairly lengthy discussion (teacher-students and students-students) about how the posed problems were the same or different, and in what ways (e.g.,  and ).]

... .

T: Well, time is up, but the first method by Snobe-kun is increasing the number of squares, and the methods by Tani-kun, Endo-kun, Kaneko-san, Suzuki-san and Tsunashima-kun involve changing the figures. Of course, the number is also changed.

[The teacher pointed to two problems like Suzuki-kun's second problem, i.e., the converse problem.]

These two interesting problems are different from the others - they give the number of matches and ask how many figures can be made.

Each problem belongs to one of three types. In the first type, the number is changed, that is, the number of squares. In the second type, the figure is changed. The third type is the converse problem.

What type of problem would you want to solve if you were to solve one of these problems?

S: Endo-kun's problem.

T: And you?

S: Endo-kun's problem.

The answer for the original problem is unique, but there is a rich variety of methods or ways of thinking that students use to find the answer. The teacher can see some cognitive development by observing individual student findings and, through discussion, how the findings of students are similar or different and classifying the results into categories.

b) *Assessment of individual concept formation*

Especially in whole-class instruction, attention should be paid to individual students; this is particularly true for teaching basic ideas in mathematics. Teachers must grasp students' thinking and deal with each one individually. In doing so, it is extremely important that teachers try to anticipate all possible ways of students' thinking and consider the methods that may be used to deal with them beforehand.

Example - Number up to 100: first grade (Matsubara 1987)

T: There are many marbles in this box. When I take a handful of marbles, how many do I have?

S1: 50 marbles.

S2: 100 marbles.

S3: 84 marbles.

T: How many marbles can you take?

S4: 40 marbles.

S5: 100 marbles.

S6: My teacher, let's take a handful!

T: O.K., please come take a turn.

[Students take 30 to 50 marbles]

T: Who took the most marbles? Please count them. How many marbles did each of you take? Please arrange them in such a way that I can understand.

[Even before I asked them to count, they began. Walking around the classroom, I observed how the students were counting. I asked for their methods, one by one.]

T: [To a student who counts one by one]

How many marbles are there?

S7: 41 marbles.

T: I am not sure whether the number of marbles is 41 or not, unless I count one more time.

T: [To a student who counts by tens, with 10 marbles in a line]

How many marbles did you take?

S8: 35 marbles?

T: How did you do it?

S8: Here, 1, 2, 3, ... 10, there are 10 marbles. Since there are three 10s, that's 30 marbles, and 5 marbles remain. In total, 35 marbles.

T: Well, you arranged the marbles in a good way.

T: [To a student who makes a pile of marbles]

What is that pile?

S9: I separated 10 marbles.

T: Please show me whether the number of marbles is really 10 or not.

S9: 1, 2, 3, ... 9. Oh, there are only 9 marbles here!

T: Please make a group in such a way you can see the marbles clearly.

. . . . . .

Finally, the teacher found five ways of counting by the students:

- Counting one by one
- Counting by piles of ten
- Counting by ten in a triangle
- Counting by five in a line, ten in two lines
- Counting by ten in a line

The teacher assesses each individual student and deals with each approach used. Therefore, the teacher could determine that all the children were ready for the place-value system.

c) *Assessment during discussion*

Discussion between students and the teacher is usually undertaken after students individually tackle a problem. In the discussion, students can see firsthand that different opinions among them exist and can then recognize concepts more deeply. The teacher's questions and observations together promote the discussion. Observation during students' work on a problem becomes an important source of information for questions.

Example - Linear equations: seventh grade (Handa 1992)

The teacher presented the following problem:

> When do the long hand and the short hand overlap each other on a clock between 1 and 2 o'clock, and 2 and 3 o'clock?

The students thought freely about this situation. Then the teacher observed students' responses by walking around and classified them roughly as follows:

- Many students made an equation
- Some students solved the problem without making an equation
- A few students drew a figure

The teacher started the discussion.

T: Please present how you solved the problem. [Intentionally, the teacher named a student who solved the problem without making an equation.]

S1: In elementary school, I had solved it by using a mathematical expression. The long hand proceeds 6° in a minute. Since the short

hand proceeds 0.5° in a minute, the long hand overtakes the short hand (6 - 0.5)° = 5.5° in a minute. Therefore, to overtake 30°, it takes 30 ÷ 5.5, namely, 60/11 minutes. The first overlap is 1 o'clock 5 5/11 minutes. I thought the same way for the next problem.

T: Are there any questions for this way of thinking? No? Then how can we solve the situation between 2 and 3 o'clock by this way? (Teacher calls on another student.)

S2: At 2 o'clock there is a 60° difference at first, 60° ÷ 5.5° or 11 minutes.... Just a minute. [She started to reduce 600/55]

S3: Since it is twice, it is 120 elevenths.

T: You said it was twice, how did you think of that?

S3: The first difference, 60°, is twice as much as 30°.

T: I see; if so, how can we calculate the time the long hand and the short hand overlap at past 3 o'clock? [Teacher calls on a different student.]

S4: Since it is three times, it is 180/11 minutes.

T: In the same way, for 4 o'clock, 5 o'clock, to 60/11, we do four times, five times....

Now, we will let students who solved the problem using other methods present theirs. [Teacher intentionally named a student who solved the problem using an equation.]

S5: I solved by making an equation. After all, it's the same.

T: You are right. If it were different, it would trouble us (laughing). What equation did you make?

... .

The teacher used two types of observation in this situation: observation during the tackling of the problems and observation during the discussion. The teacher's objective was reflected in selecting which students to respond. The teacher wanted the students, as a class, to consider the solution methods from primitive to more sophisticated ones.

(2) Assessment After Lessons

Essentially, there should be consistency in assessment from during the lesson to after the lesson. An example of this is given below. The evaluation is carried out according to students' ways of mathematical thinking, which were found on their worksheets.

Example - Manipulating on mathematical expressions: seventh grade (Ohta 1990)

The teacher used a "number game" to introduce manipulation on mathematical expressions in a two-hour lesson as follows:

Double a number that each student selects, add 2, and multiply the result by 5. Students were then asked to find the first number when the last number was given. On a part of the lesson with the teacher's analysis is as follows:

One student grasped the structure by using a figure, and another student (S1) grasped a more abstract explanation, shown by the student's own explanation as follows:

S1: It is represented as $(x \times 2 + 2) \times 5$.

Skip + 2, and multiply

It becomes $x \times 2 \times 5$, then $x \times 10$.

There remains 2×5, then 10

Therefore, subtract 10 and divide by 10.

To this explanation, some students nodded their heads yes, but most students put their heads a little to one side (indicating no). In general, there was no atmosphere

indicating that students understood. Another student (S2) stood up to explain as follows:

S2: Suppose the first number as 2

[But he also used ○ and ⌓ on the chalkboard.]

ten times ○

(○) x 2 = ○○ } x 5, therefore ⌓⌓⌓ ⌓⌓⌓ ⌓⌓⌓ ⌓

If I add 2 to this, as ⌓⌓ }

saying aloud in numbers, four, six, thirty.

From this number, I can find the answer by subtracting 10 and dividing by 10.

S: Excellent! We understand well!

S2 supposed the first number to be 2. Although depending on this number at first, gradually S2 became detached from it and gained insight into the relation. This can be seen from the fact the S2 said "ten times" without figures. After students gained insight into the structure in this way, many students recognized the explanation by using letters.

After about one month, the teacher used the same type of problem on the semester examination as follows:

Number game:

1. Choose a positive number and add 3.
2. Multiply the result of (1) by 2.
3. Subtract 3 from the result of (2).
4. Multiply the result of (3) by 5.

If the result of (4) is 155, what is the original number?

How did you find it?

Explain how to find it.

The teacher analyzed students' explanations and found seven types of meaningful responses concerning the use of letters as follows:

- Uses a literal expression (roughly) and tries to explain by transformation,
- Explains by using numbers, but does not depend on the numbers from the viewpoint of content,
- Explains by depending on numbers but cannot detach from numbers,
- Finds a relation inductively,
- Explains by figures,
- Explains by language,
- Finds by the reverse process.

The teacher evaluated each student according to these categories. Usually, it is difficult to carry out this type of analysis on a semester examination, since there is too little time. But if it is carried out, the result is useful not only for summative evaluation but also for formative evaluation in the long run.

IV. TEACHING-EVALUATING RESEARCH TO ASSESS HIGHER ORDER THINKING

In addition to the assessment of students' different solution methods, long-term research projects assessing higher-order thinking have been carried out collaboratively by many teachers and researchers organized by the National Institute for Educational Research (NIER). In teaching for higher-order thinking objectives, such as ways of mathematical thinking, creative thinking, attitudes toward-and interest in-mathematics, teachers must devise new problem situations in which a variety of students' thinking is possible and, indeed, expected. We shall describe two research projects.

1. The Open-End Approach To Teaching Mathematics

(1) Rationale

In both Japan and the United States, problems traditionally used in teaching mathematics have the common feature that there is one and only one answer, and it is predetermined. These problems are so well formulated that answers are either correct or incorrect (including incomplete ones). Such problems may be called "closed" or "complete" problems. In contrast, problems that are formulated to have multiple correct answers are "incomplete," "open end," or "open ended" (Becker and Shimada 1993). This approach may have students find one, several, or many correct answers to one problem, use one, several, or many different methods to arrive at their answers, or pose or formulate problems of their own. In such instances, it is possible for students to learn new things in the process by combining their own knowledge, skills, or ways of thinking that have been previously learned (Becker and Shimada 1993).

The "open end" approach derives from the work of Japanese researchers and teachers in devising approaches to evaluating students' achievement of higher-order objectives from their study of mathematics. As is well known, mathematics teaching usually centers on knowledge (skills, concepts, principles, or laws) presented in step-by-step fashion in the curriculum. In isolation, each is not important; however, it is expected that they will be integrated into each student's abilities and attitudes, being internalized and thereby becoming part of the intellectual organization within the mind of each student. Thus, each is an integral component of the whole, and the essential point is that they all be integrated into the intellectual make-up of each student.

In order to know the extent to which a student achieves higher-order thinking, it is necessary to observe how the student uses what is learned in a concrete situation. This

requires the concrete situation to be a natural one, not an artificial one instituted by others for the purpose of evaluation. At the same time, most paper-and-pencil tests used for evaluation use the closed type of problems, whereby all the mathematical conditions needed for solution are provided and students need only apply their knowledge, skills, and so on by retrieving what is appropriate from their repertoire of previous learning according to the problem conditions. Evaluation, therefore, cannot go beyond checking students' achievement in terms of previous learning and its application.

(2) The Open-end approach

In order to implement the "open-end approach" in teaching mathematics, Japanese researchers and teachers organized a teaching-evaluating research program using several carefully developed mathematics problems. The overall aims were to answer the following questions (Becker and Shimada 1993):

- Can mathematics problems be developed such that when students solve them, they exhibit behavior(s) reflecting higher-order thinking?
- How are these behaviors exhibited by students in their problem solving, and how are they related to students' achievement as measured by ordinary paper-and-pencil tests?
- Can the behaviors of students, thought to be higher-order thinking, be further ripened or developed?

The theoretical framework and rationale for this work is given in Becker and Shimada (1993). In the project, three types of open-end problems were devised:

- How to find rules or relations
- How to classify
- How to measure

Examples of Problems

The following problems are examples of problems used in the work:

Example 1 (Many different rules/relations)

The following table shows the record of five baseball teams (A, B, C, D, and E). Among the figures in his table, there are certain regularities or rules or relations. Find as many of these as you can and write them down.

Team	Games	Wins	Losses	Draws	Winning Ratio	Games Behind
A	25	16	7	2	0.696	---
B	21	11	8	2	0.579	3.0
C	22	9	9	4	0.500	1.5
D	22	8	13	1	0.381	2.5
E	22	6	13	3	0.316	1.0

Example 2 (Many different way of classification)

There are several solids as follows:

A

B

C

D

E

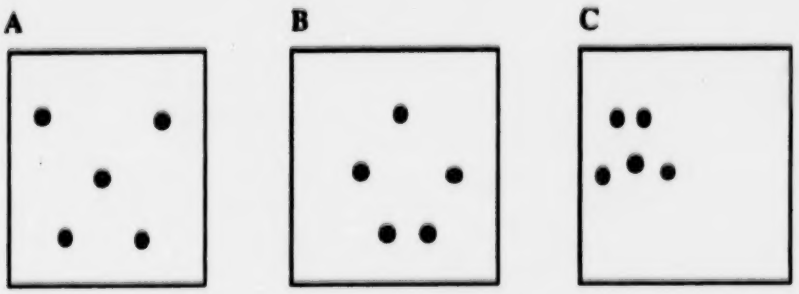
F

G

H

Choose the solid(s) that share(s) the same characteristic(s) with the solid B and write down the characteristic(s).

Example 3 (Many different ways of measuring)



Three students, A, B, and C, throw five marbles that come to rest as in the figures above. In this game, the student with the smallest scatter of marbles is the winner. To determine the winner, we will need to have some numerical way of measuring the scatter of marbles.

- Think about this situation from different points of view and write down different ways of indicating the degree of scattering.
- Which way do you think is the best one?

The examples above are intended only to illustrate, but certainly not represent, the full range and variety of problems developed by Japanese researchers and teachers who worked cooperatively in the teaching-evaluation research project (see Becker and Shimada 1993) for numerous other problems used in both the research project and Japanese classrooms.

In the teaching of these problems, the teacher organizes the lesson according to the following scheme (Becker et al. 1990):

i. Introducing the problem

The teacher presents or poses the problem on the OHP, blackboard, or poster.

ii. Understanding the problem

The teacher ensures that students know what is expected before they begin work.

iii. Problem solving by students

Students are given a worksheet with the problem written on it and work individually and/or in small groups. Emphasis is placed on appealing to students' natural ways of thinking - "drawing out" a variety of responses. The teacher moves among the students, purposefully scanning their work and selecting the approaches or answers that will be discussed with the whole class.

iv. Comparing and discussing approaches or answers

Individual students (or groups) write their approaches or answers on the blackboard (or OHP) for all students to see. Then the teacher guides a comparison and discussion of the responses and groups or classifies them according to the same mathematical ideas for discussion of their mathematical quality.

v. Summing up the lesson

The teacher plays an important, even crucial, role in "pulling together" the outcomes of the discussion as it relates to the lesson's objective.

It should be clear that the problems themselves and the careful organization and management of the lesson are of crucial importance. Overall, the classroom activities are structured to help students

- to "mathematize" situations appropriately;
- to find mathematical rules or relations by making good use of their knowledge and skills;
- to solve the problems in a variety of ways;
- to check their results;

while

- seeing other student's discoveries and methods;
- comparing and examining the variety of different ideas of students;
- modifying and further developing their own ideas accordingly.

The "open-end approach" can be seen to relate to certain goals and priorities set forth in NCTM's Curriculum and Evaluation Standards for School Mathematics (1989).

(3) Assessment

The heart of the "open-end approach" has the teacher using problems that are carefully developed, that are known to work well with students, and that furnish a context in which each student can use her or his own natural ways of thinking, thereby generating a variety of approaches to solving the problems. To assess students' learning, (a) as students work on the problems and during discussion, the teacher has an opportunity to observe their mathematical behaviors and thinking, and (b) the teacher can collect students' worksheets to examine as another source of information. Though the approach may not appear easy for assessing students' learning, it has been demonstrated to work well.

As mentioned earlier, the teacher prepares in advance of the lesson, a listing of students' possible responses (e.g., ways of thinking about the problem and answers). These can be classified and arranged in an order, one by one, according to their mathematical quality. Then during the lesson, the teacher can observe students' actual responses and record them in a chart devised by the teacher. Students' achievement can then be evaluated using this chart from the following points of view:

i. Fluency

How many different answers or approaches to finding the answers did the student produce?

If a student's (or group's) response is correct from a certain point of view, the teacher may give the student (or group) one point. The total of these points is then the "total number of responses," which can be regarded as an indication of the fluency of students' mathematical thinking.

ii. Flexibility

How many different mathematical ideas were discovered by the student? Correct solutions or approaches to getting the answer may be partitioned into several categories, and if, for example, two solution (or approaches) have the same mathematical idea in common, they should be included in the same category. The number of categories may be called the "number of positive responses." Such a number can be regarded as a measure of the flexibility of students' mathematical thinking-the larger the number, the greater the flexibility.

iii. Originality

To what extent are the students' ideas original?

If a student (or group) develops a unique idea not found by other students (or groups), the originality of the idea should be given a high evaluation. Among expected responses, there may be several levels of mathematical quality, from low to high: low score for lower quality, high score for higher quality. The total number of such responses may be called the "number of positive responses with weight," and it indicates the originality of a student's (or group's) idea.

'Fluency' and 'flexibility' are quantitative assessments ("How many?"), whereas 'originality' is a qualitative assessment ("How innovative?"). It is important that we point out that in the Japanese research, students with substantial experience in the open-end approach got higher scores in terms of flexibility and originality then students with no such experience (Becker and Shimada 1993).

But there is still another dimension to assessment in connection with the open-end approach. We refer to the degree of elegance in students' expression of thinking. No doubt some students will write solutions in unclear, ambiguous ways, whereas others may do so in a clear, simple, and even elegant manner. The latter is preferred, especially when students use mathematical (algebraic) notation to express their thinking. Objectively evaluating the degree of elegance, however, may not be easy to do. Nevertheless, it has potential as part of the assessment of students' learning.

2. The Developmental Treatment of Mathematics Problems

(1) Rationale

At the end of the research on the "open-end approach" mentioned above, some research issues remained to be resolved. For example, Hashimoto and Tsubota (1977) mention that future study should focus on two situations other than the open-end one; namely, the open-beginning situation and the open-middle situation. Accordingly, research on the development treatment of mathematics problems started in 1978. The research style was the same as for the "open end approach." Though the aim of the research was the development of an evaluation method for higher objectives of mathematics education, teaching based on the idea of the open-beginning situation was addressed first.

(2) The Developmental Treatment

A year later, the teaching of the developmental treatment of mathematics problems was defined (Sawada et al. 1980) as

> teaching focused on learning activities with students formulating new problems from a given problem using generalization, analogy, the idea of converse, etc., and then solving them by themselves. (p. 206)

The research was conducted from 1978 to 1986 at all school levels. The work involved about fifty teachers and researchers working collaboratively. During this period, the teaching practice, mathematical topics, and evaluation methods were studied. In particular, a lot of problems made up by students from the first to the twelfth grade were collected (Nagaski and Hashimoto 1985).

Teaching practice

In the developmental treatment of mathematics problems, the teacher organizes the lesson in the following sequence (Hashimoto and Sawada 1984; Takeuchi and Sawada 1985):

- Solve a given problem,

- Discuss the methods of, and the solution to, the problem,
- Formulate new problems by changing parts of the given problem,
- Propose new problems to a whole class,
- Discuss some of the new problems and classify them,
- Solve common problems selected by the teacher or students,
- Solve students' own problems.

A good example can be seen under "Number of Matches" in the section "Assessment Practice" on pages 9 - 19. One research issue was how to deal with the third stage above, namely, how to encourage students to formulate new problems. After all the research members had tackled this issue in various ways, some very simple instructions were formed (Nagasaki 1981).

Teacher's first instruction:

- Let's formulate new problems on the basis of a given problem,
- Let's formulate problems similar to the given problem.

For students who are not able to formulate or pose problems according to the above, the following instructions may be used:

- Which parts of the given problem can be changed?
- Let's change the parts of the given problem that can be changed.

In addition, it is effective to show students problems that were formulated by other students.

(3) Problems Formulated by Students

In this way, students made up several problems. During experimental lessons, the teacher observed students' learning to get and use information in the last four stages above. After the lesson, the teacher collected all worksheets and classified the problems. The given

problems and problems formulated by students are shown by Takeuchi and Sawada (1985) as follows:

a) Written problem on addition (First graders)

Given problem: There were 50 pencils. I brought 30 more pencils. Altogether, how many pencils are there?

Problems formulated by students:

- There are 50 sheets of drawing paper. I was given 30 more sheets of drawing paper. Altogether, how may sheets of drawing paper are there?
- There were 80 pencils. I was given 20 pencils. Altogether, how may pencils are there?
- There were 45 pencils. I was given 12 pencils. Altogether, how many pencils are there?
- There were 50 persons in a bus. At Tenjin bus stop, 30 persons got off. Now, how many persons ar there in the bus?
- There are 45 sheets of folding paper. I used 15 sheets. How many sheets of folding paper are left?

b) Number of "Go" stones of the game of "Go" (Fourth graders)

Given problem: I put "Go" stones on all sides of a square. When I put 5 "Go" stones on a side, how many "Go" stones are there altogether?

Problems formulated by students:

- I put "Go" stones on all sides of a square. When I put 10 "Go" stones on a side, how many "Go" stones are there altogether?
- I put "Go" stones on all sides of a regular pentagon. When I put 4 "Go" stones on a side, how many "Go" stones are there altogether?

- I put coins on four sides of a square. Altogether, there are 64 coins. How many coins are there on a side?
- I put "Go" stones in three lines on a square "Go" board. There are 91 stones on the outside. Altogether, how many "Go" stones are there?
- I put marbles on two sides of a regular hexagon. There are 91 marbles on two sides. How many marbles are there on the six sides?

c) Written problem on linear equations (Seventh graders)

Given problem: When the sum of three consecutive odd numbers is 177, what are the three numbers?

Problems formulated by students:

- When the sum of five consecutive odd numbers is 35, what are the five numbers?
- When the sum of four consecutive even numbers is 300, what are the four numbers?
- When the sum of five consecutive numbers is 105, what are the five numbers?
- When the product of three consecutive integers is 1716, what are the three numbers?
- There are several odd consecutive numbers starting with 57. When more consecutive numbers are added, the sum is 177. How may more consecutive numbers were added?
- A younger brother is two years younger than his older brother, and the older one is fifteen years younger than his father. The sum of their ages is 91. How old is each? [Note: The problem situation is not realistic, as may be common with children.]

d) Range of solutions of quadratic functions (Tenth graders)

Given problem: One of the solutions of the quadratic equation $x^2+4x+a=0$ exists between 0 and 1. What is the range of values for a?

Problems formulated by students:

- One of the solutions of the quadratic equation $x^2-4x+a=0$ exists between 2 and 4. What is the range of values for a?
- Two solutions of the quadratic equation $x^2+6x+a=0$ exist between -1 and 5. What is the range of values for a?
- One of the solutions of the quadratic equation $x^2-ax+3=0$ exists between 0 and 1. What is the range of values for a?
- One of the solutions of the quadratic equation $x^2+4x+a+b=0$ exists between 0 and 1. What is the range of values for a and b?
- The solution set of the quadratic inequality $x^2+4x+a>0$ has an element 0. What is the range of values for a? (Similar types: the quadratic equation is changed to a linear equation, cubic equation, fractional equation, and irrational equation.)
- The graphs of $y=x^2+2x+3$ and $y=-x^2+3x+a$ touch each other. What is the range of values for a?

(4) Assessment

The approach to assessment in this work was similar to that of the "open-end approach" to assess students' learning: (*a*) as students solve a given problem and discuss it, the teacher has an opportunity to observe their mathematical behavior and thinking; (*b*) as students formulate their own problems and discuss and solve them, the teacher has another

opportunity to observe their mathematical behavior and thinking; (*c*) the teacher can collect students' worksheets and examine them as still another source of information.

Before teaching the lessons, teachers prepared a list of problems that students might formulate. They need this list to analyze the problems and to organize a discussion of the problems. After the classroom lesson, the teacher again tried to analyze and classify students' problems. Their problems could then be evaluated from three points of view as follows (Nagasaki et al. 1983):

a) By the number of problems

How many problems was the student able to formulate? In our research, the average number of problems was about three.

b) By how the students formulated their problems

- Students changed only a number(s)
- Students changed only an object(s)
- Students changed a number(s) and an object(s)
- Students formulated a new problem by analogy
- Students formulated a new problem by using the converse of the given problem
- Students formulated a new problem by using several problems
- Others
- Wrong problems

c) By the direction of development from the given problem

This view reflects ways of thinking mathematically. The following categories are used to classify problems:

- Generalization

(1) The same structure is used

(2) The geometrical figure is changed, but the structure is the same

(3) The relation is generalized.

- Analogy

(1) The structure is the same, but the situation is different

(2) The operation is changed

(3) The geometrical figure is changed as well as the structure

(4) The dimension is changed

(5) The structure is changed, but part of the hypothesis is retained

- Converse

(1) The hypothesis and conclusion are exchanged

- Compound

(1) Another (other) condition(s) is(are) added

- Others

One example of analysis from the second category is given here (Hashimoto and Sakai 1983 and Hashimoto and Sawada 1984):

Original problem

Take a point *P* on the diagonal AC in parallelogram ABCD. Through *P* draw a line EG parallel to AD and a line HF parallel to A*P* as in Figure 1. Prove that *PH*: *PF* = *PE*: PG.

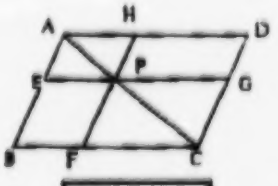


Figure 1

Analysis of problems made by students:

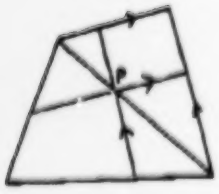


Figure 6

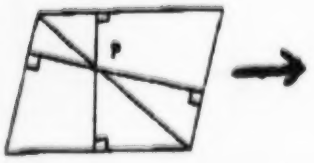


Figure 2

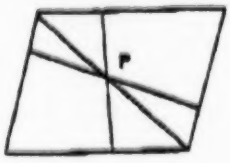
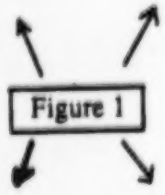


Figure 3



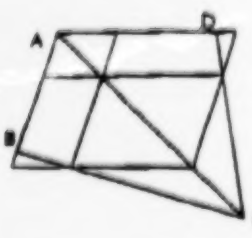


Figure 7

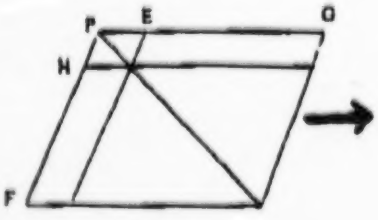


Figure 4

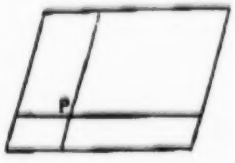


Figure 5

Figure 2-7 are variations of figure 1. The problems corresponding to these figures were made by changing parts of the original problem or by using the converse of the given problem. The explanation of the problems is as follows.

Figure 2: Draw perpendicular lines instead of parallel lines.

Figure 3: Draw arbitrary lines instead of parallel lines. (Even in this case, the conclusion that $PH : PF = PE : PG$ is satisfied. Then one of the generalizations is satisfied, and students can learn the property between parallel lines and proportion.)

Figure 4: Change how to take a point. For example, take a point on the extension of the diagonal.

Figure 5: Change how to take a point. For example, take any point inside the parallelogram.

(The conclusion is not satisfied in this case. Students can easily find a counterexample.)

Figure 6: Change the shape.

(Since the conclusion is not satisfied in this case, one of the generalizations is not satisfied.)

Figure 7: Consider the converse of the given problem.

(The conclusion is not satisfied in this case, since we can find a counterexample. In reality, students could not find it.)

The first three categories above were proposed for formative and summative assessment. Though the teachers might usually have these categories in mind in classroom teaching, problems formulated by students were usually classified according to the first and second ones, especially in summative assessment. For the third, the categorization seemed too sophisticated; the view was a difficult one with which to deal.

In summary, almost all the teachers thought that the first category was simple. Since formulating problems was a new experience for both students and teachers, only the first category was thought to be important. The developmental treatment became familiar to many teachers as a means for improving instruction, not as a means of assessment. However, since we think that assessment should be integrated into classroom teaching, many teachers now use the open-end approach and the developmental treatment as an initial step toward improving assessment.

V. CONCLUSION

In Japan, assessment (evaluation) in classroom teaching cannot be considered apart from classroom lessons. Japanese mathematics lessons involve whole-class instruction, have several stages, and respect the importance of lesson plans and lesson records.

Even though classroom lessons involve mainly whole-class instruction, mathematics educators also need to cope with individual students' activities or needs. Furthermore, in order to foster students' thinking mathematically, it is important to allow them to think freely. Therefore, it is expected - and is crucially important - that teachers use students' different ways of thinking in their lessons.

In the main, classroom lessons using students' ways of thinking and the importance of formative evaluation are appreciated by teachers in general, but many also tend to depend on summative evaluation using paper-and-pencil tests. This is due primarily to constraints placed on evaluation by the dominant role of the entrance examinations in Japanese education.

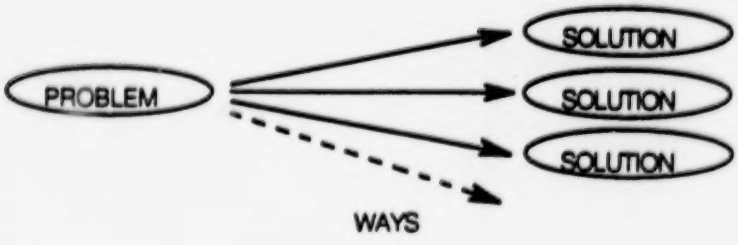
At the same time, classroom practice and assessment that use students' natural ways of thinking are used by teachers. It is commonly accepted that even if a problem has only one solution, there may be several ways to find the solution. In this situation, the observation of students during lessons and analyses of students' worksheets after the lessons are important vehicles for carrying out assessment. Indeed, this is the starting point to using students' different ways of thinking. But in addition to using several ways to solve a problem with a unique answer, the open-end approach, in which teachers use a problem that has different correct solutions, and the developmental treatment, during which students formulate or pose problems of their own, are also devised for using students' various ways of thinking. These three main modes of teaching are illustrated in figure 5.2 (Shimada 1976).

1. ONE PROBLEM . . . . . ONE SOLUTION (ANSWER)

WAYS

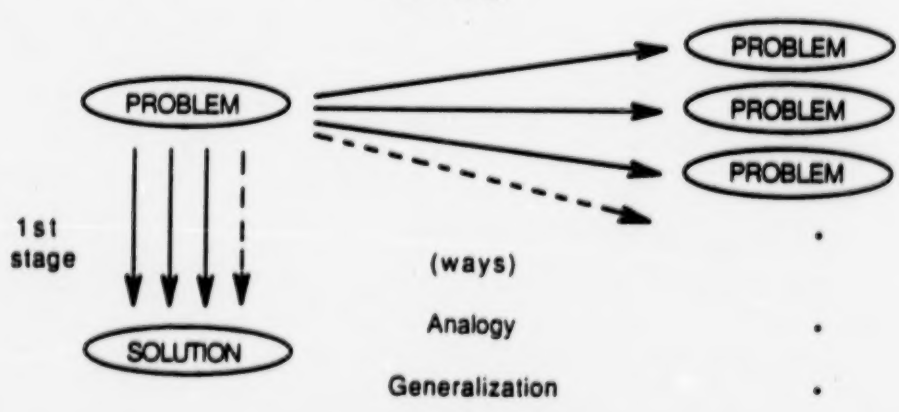
(Process is open)

2. ONE PROBLEM (OPEN ENDED).... SEVERAL OR MANY SOLUTIONS (ANSWERS)



(End products are open)

3. ONE PROBLEM . . . . . SEVERAL PROBLEMS
("From problem to problem . . . . . the developmental approach")



(Ways to develop are open)

Openness in Mathematics Education

Fig. 5.2

In this context, we have discussed assessing higher objectives in mathematics education. In order to assess them, it is crucially important that students be given opportunities to express their thinking freely and openly, and the ways in which students express their ideas should be assessed and evaluated.

This view has had some effect on mathematics education in Japan. When teachers recognize students' cognitive tendencies, it helps them to improve their instruction and thereby contribute to teacher development. Furthermore, curriculum improvement from the "grass roots" level also becomes possible (Howson et al. 1981).

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